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A 3D Two-Phase Numerical Model for Sediment Transport

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Abstract

We have developed a three dimensional numerical model based on the two-phase equations to study the bed-load transport Chauchat and Médale (2009). We have considered two formulations of the model based on a two fluid or a single mixed-fluid description. The governing equations are discretized by a finite element method and a penalisation method is introduced to cope with the incompressibility constraint whereas a regularisation technique is used to deal with the visco-plastic behaviour of the granular phase. The accuracy and efficiency of the numerical models have been compared with the analytical solution of Ouriemi et al. (2009a). It turns out that one must takes a smaller regularisation parameter (one order of magnitude) in the mixed-model than in the two-fluid one for a comparable accuracy. Using an Arc Length Continuation algorithm coupled with this model we have investigated the evolution of the solution in terms of the height of the flowing granular layer and the particle flux against the longitudinal pressure gradient. The results of these simulations are in good agreement with the analytical solution in the 2D case and three-dimensional computations have been carried out in a square and circular cross-section ducts showing that the effect of the geometry is non-trivial.

Introduction

The transport of sediment or more generally the transport of particles by a fluid flow is a problem of major importance in geophysical flows such as coastal or river morphodynamic or in industrial flows with the hydrate or sand issues in oil production and granular transport in food or pharmaceutical industries. This problem has been extensively studied in the literature since the middle of the twentieth century but poorly understood actually (Einstein 1942; Meyer-Peter and Muller 1948; Bagnold 1956; Yalin 1963).

Recently, Ouriemi et al. (2009a) have proposed a two-phase model describing the bed-load transport in laminar flows that allows to incorporate more physics than in previous modelling based on particle flux or erosion deposition approaches. This two-phase model is based on a Newtonian rheology for the fluid phase and a frictional rheology for the particulate phase ($\mu(I)$ Forterre and Pouliquen (2008) or Coulomb friction) while the fluid-particle interaction is assumed to follow a Darcy law. This approach allows to predict the threshold of motion for the particle phase and give a description of the flow of inside the flowing granular layer. Away from the threshold of motion, a simpler analytical model for the particle flux is obtained which gives a quite satisfactory description of experimental observations of bed-load transport in pipe flows Ouriemi et al. (2009a).

Based on this theoretical model we have developed a 3D numerical model that allows to simulate bed-load transport in 2D or 3D configurations (Chauchat and Médale 2009). It is restricted to the cases where the granular bed does not change its shape in the course of time, consequently ripples and dunes formation are beyond the scope of this paper. We consider a mixed-fluid and a two-fluid formulation, the models equations are discretized by a finite element method and a penalisation method is used to impose the incompressibility constraint. The granular rheology is analoguous to a viscoplastic behaviour especially by the existence of a yield stress with the particularity that this yield stress depends on the depth inside the granular layer. A regularisation technique is used to deal with the viscoplastic rheology and is shown to give satisfactory results.

Two-phase model

The present model is based on Jackson (2000, 1997) averaged equations using the closures developed by Ouriemi et al. (2009a). These equations are summarized hereafter in dimensionless form using the following scaling: the length is scaled by H, the channel height (see figure 1), and the stresses is scaled by $\Delta\rho g H$, and therefore the time is scaled by $\eta/\Delta\rho g H$ where $\Delta\rho = \rho_p - \rho_f$. The problem is expressed in terms of the solid volume fraction ϕ, the mixture velocity $\overrightarrow{u^m}$ and the particulate velocity $\overrightarrow{u^p}$.

Two-fluid model

$$
\begin{aligned}
\nabla.\left(\overrightarrow{u^m}\right) &= 0 \\
\nabla.\left(\overrightarrow{u^p}\right) &= 0 \\
Ga\frac{H^3}{d^3}\frac{D\overrightarrow{u^m}}{Dt} &= -\nabla p^f + \nabla.\left(\frac{\eta_e}{\eta}\overline{\overline{\dot{\gamma}^m}}\right) \\
&\quad -\frac{H^2}{K}\left(\overrightarrow{u^m} - \overrightarrow{u^p}\right) + \frac{\rho_f \overrightarrow{g}}{\Delta\rho \parallel \overrightarrow{g} \parallel} \\
Ga\frac{H^3}{d^3}R_\rho\phi\frac{D\overrightarrow{u^p}}{Dt} &= -\nabla p^p - \phi\nabla p^f \\
&\quad +\nabla.\left(\frac{\eta_p}{\eta}\overline{\overline{\dot{\gamma}^p}}\right) + \phi\nabla.\left(\frac{\eta_e}{\eta}\overline{\overline{\dot{\gamma}^m}}\right) \\
&\quad +\frac{(1-\phi)H^2}{K}\left(\overrightarrow{u^m} - \overrightarrow{u^p}\right) + \frac{\phi\overrightarrow{g}}{\parallel \overrightarrow{g} \parallel}
\end{aligned}
\tag{1}
$$

In these equations, $\overline{\overline{\dot{\gamma}^k}} = \nabla\overrightarrow{u^k} + (\nabla\overrightarrow{u^k})^T$ with $k = m$ or $k = p$, $R_\rho = \rho_f/\rho_p$ represents the density ratio and $Ga = d^3\rho_f\Delta\rho g/\eta^2$ is the Galileo number where d is the particle diameter. The Galileo number is a Reynolds number based on the settling velocity of particles.

Mixed-fluid model

$$
\begin{aligned}
\nabla.\left(\overrightarrow{u^m}\right) &= 0 \\
Ga\tfrac{H^3}{d^3}(1+R_\rho)\frac{D\overrightarrow{u^m}}{Dt} &= -\nabla p^f - \nabla p^p + \frac{\rho_m \overrightarrow{g}}{\Delta\rho \parallel \overrightarrow{g} \parallel} \\
&\quad +\nabla.\left(\frac{\eta_e + \eta_p}{\eta}\overline{\overline{\dot{\gamma}^m}}\right)
\end{aligned}
\tag{2}
$$

To close these equations we need to prescribe the effective fluid viscosity $\eta_e = \eta(1 + 5/2\phi)$ (Einstein 1906), the particulate viscosity $\eta_p = \mu_s p^p / \parallel \overline{\overline{\dot{\gamma}^p}} \parallel$ (Jop et al. 2006) with μ_s the internal friction coefficient and the permeability $K = \frac{(1-\phi)^3 d^2}{k_{CK}\phi^2}$ with $k_{CK} \approx 180$ (Happel and Brenner 1973).

Numerical model

The Finite Element Method (FEM) is based on the discretisation of the variational formulation associated with equations (1) for the two-fluid model and on equation (2) for the mixed-fluid model.

The two-fluid formulation is based on the solution of the system (1) in a weakly coupled way. The algorithm associated with the mixed-fluid formulation is based on the solution of the system (2). The non-linearities in the governing equations are solved by a Newton-Raphson algorithm by taking the first variation of the variational formulations.

The specific issue raised by the previous two-phase model lie in the calculation of the frictional stress. Following (Jop et al. 2006), the frictional stress can be written as

$$\tau^p = \eta_p \overline{\overline{\dot{\gamma}^p}},$$

with $\eta_p = \mu_s p^p / \parallel \overline{\overline{\dot{\gamma}^p}} \parallel$. The particulate viscosity diverges as the particulate shear rate tends toward zero (*i.e.:* in the static zone) raises obvious numerical problems. The basic idea to overcome this issue consists in regularizing the viscosity by adding a small quantity (λ) to the denominator of the particulate viscosity $\eta_p = \mu_s p^p / (\parallel \overline{\overline{\dot{\gamma}^p}} \parallel + \lambda)$ then the divergence is controlled by this parameter and the viscosity is kept finite. In other words, the static zone in the frictional rheology is replaced by a very viscous zone. In the next section of this paper we will discuss the influence of the value of the parameter λ on the model solution.

In our implementation, we use piecewise quadratic polynomial approximation for the velocity and piecewise linear discontinuous approximation for the pressure. In the computations, we have employed a 27-nodes hexahedra element (H27) for the velocities. The incompressibility constraint is solved by a penalisation method. The code is developed with the PETSc library Balay et al. (2001, 2004, 1997) which provides several parallel iterative and direct solvers. As we use a penalisation method to cope with the incompressibility constraint, all the algebraic systems have been solved by the MUMPS direct solver Amestoy et al. (2000, 2001, 2006) with a penalty parameter set to 10^9 for all the simulations presented in this paper.

Results

We present the results of the previous two-phase numerical model applied to the flow of a Newtonian fluid over a granular bed. First we compare the results of the numerical model with the analytical solution of Ouriemi et al.

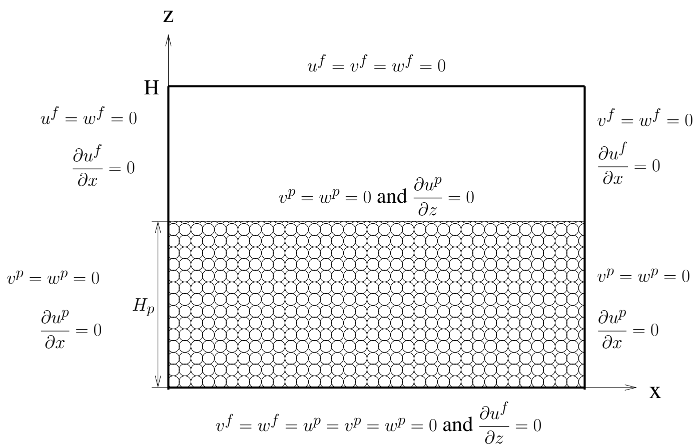


Figure 1: Sketch of the flow of a Newtonian fluid over a granular bed.

(2009a) for the bed-load transport in laminar shearing flows to validate quantitatively the two formulations of the two-phase flow model. We have also looked at the computational efficiency of the numerical model associated with both formulations.

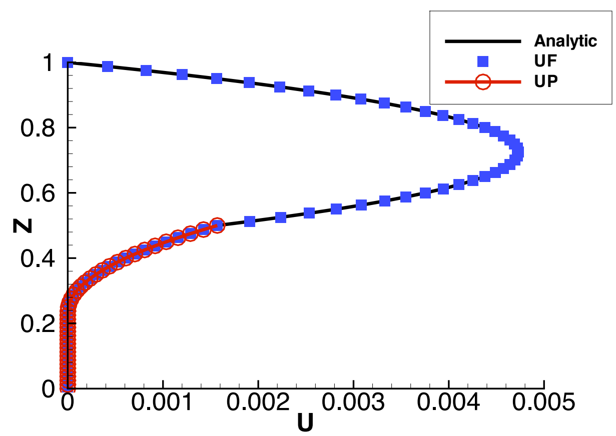
The sketch of the problem and boundary conditions are given in figure 1. The lower half of the domain is filled with particles at $\phi = 0.55$ immersed in a fluid and the upper part is filled with pure fluid ($\phi = 0$). Therefore in this problem the values of the dimensionless numbers are: $Re = 2\ 10^{-2}$, $Ga = 11$, $R_\rho = 0.4$ and $d/H = 30$. The regularisation parameter is set to $\lambda = 10^{-6}\ s^{-1}$ when its value is not mentioned in the figure captions. We solved by FEM the two formulations of the two-phase flow model for a 6x1x40 mesh with a requested absolute residual lower than 10^{-11} per degree of freedom.

Figures 2a and 2b present the comparison of the horizontal velocity profiles obtained for the two formulations of the two-phase flow model (two-fluid and mixed-fluid) by numerical simulations compared with the analytical solution proposed by Ouriemi et al. (2009a). In both figures the black solid line represents the analytical solution. The good agreement between the numerical solution and the analytical one gives a first qualitaitve validation of the FEM model for the bed-load transport.

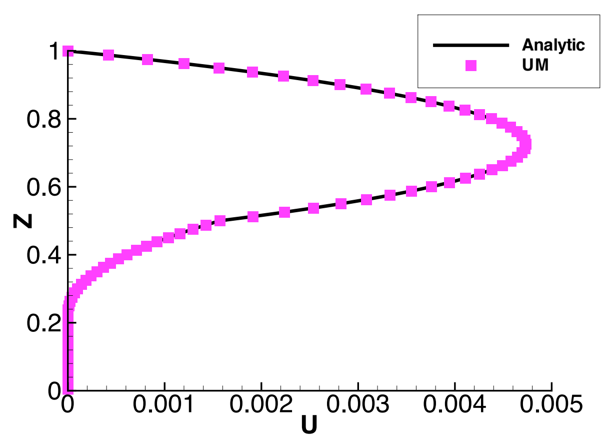
The spatial convergence analysis for both formulations (see figure 3) shows that the regularisation of the granular rheology reduces the spatial order of convergence to order one whereas a second order is theoretically hopped. Also, to reach the same accuracy with both formulations we observe that the regularisation for the mixed-fluid model must be decreased by one order of magnitude compared with the two-fluid one.

We now apply the model to three-dimensional configurations: a square and a circular cross-section ducts, with the same values of the dimensionless numbers $Re = 2\,10^{-2}$, $Ga = 11$, $R_\rho = 0.4$, $d/H = 30$ and $Bn = 2\ 10^4$. We have performed simulations on the half of the domain in the transverse direction for obvious symmetry reasons.

Figure 4 shows the velocity profile in a cross section of the ducts. The contour colors represent the x-velocity of the mixture (u^m). The horizontal thick solid line at $z = 0.5$ represents the position of the granular bed. The fluid and the mixture are sheared in both z and y directions inducing an increase in the friction compared with the two-dimensional case. Due to this shear increase the velocity is lower than in the two-dimensional case. Figure 5 shows the velocity profiles of the fluid phase velocity in blue and the particulate phase velocity in red (an offset of 10^{-3} has been added to make the particulate phase velocity visible) obtained with the two-fluid model. These results illustrate the good behaviour of the numerical model for three-dimensional flow configurations.



(a) Two-fluid model



(b) Mixed-fluid model

Figure 2: Longitudinal velocity profiles for the flow of a Newtonian fluid over a granular bed between two infinite parallel planes obtained by numerical simulations for a) the two-fluid model and b) the mixed-fluid model compared with the analytical solution of Ouriemi et al. (2009a).

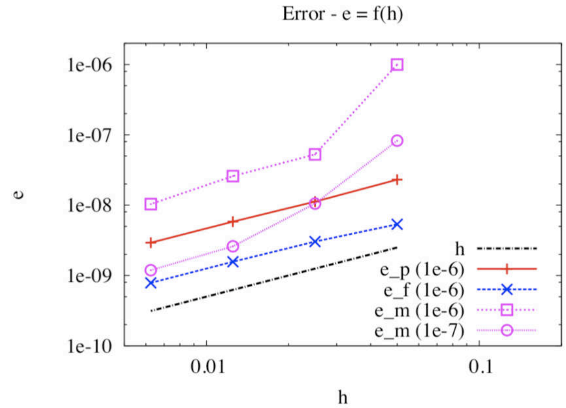
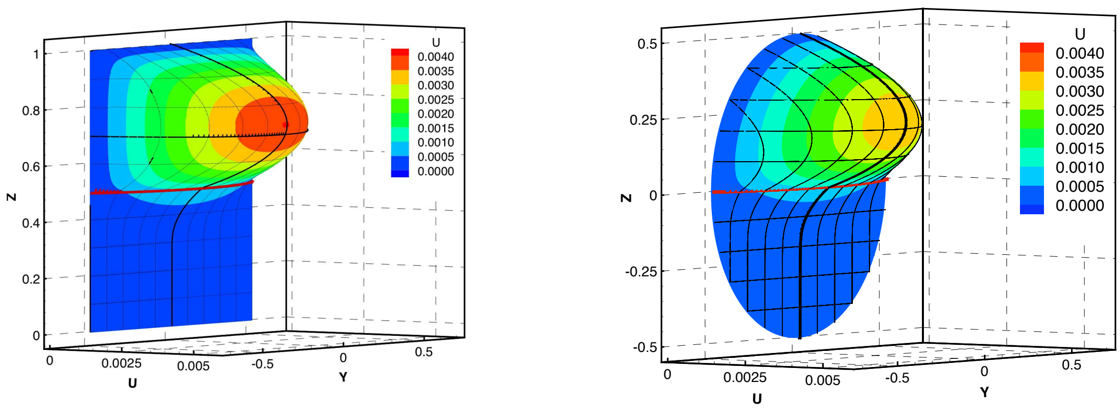


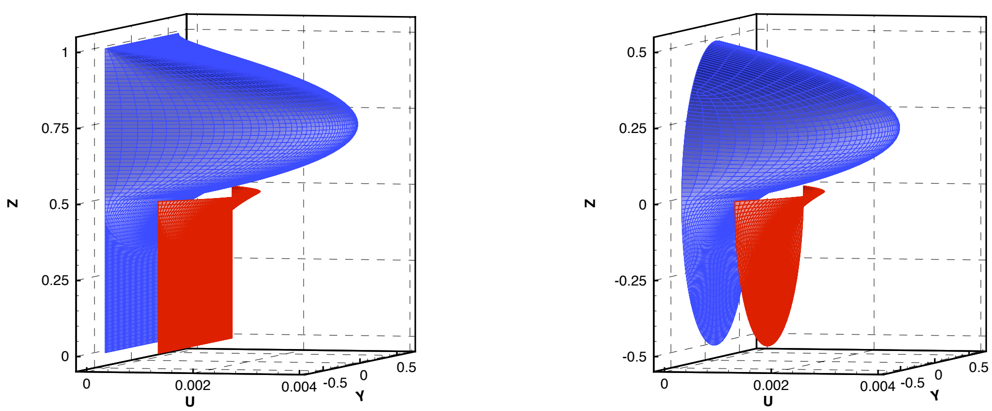
Figure 3: RMS Error against analytical solution for the flow of a Newtonian fluid over a moving granular bed between two infinite parallel planes: e_p and e_f stands for the particulate phase and the fluid phase error respectively for the two-fluid model whereas e_m designates the mixture velocity error for the mixed-fluid model. The value in brackets is the value of the regularisation parameter λ. The RMS error is defined as: $e = \frac{1}{N}(\sum_{i=1}^{N}(U_i - U_i^{ana})^2)^{1/2}$ where N is the number of nodes in the mesh.



(a) Square duct

(b) Cylindrical duct

Figure 4: Velocity profile obtained by numerical simulations with the mixed-fluid model for a) the square cross-section duct (6x20x40) and b) the circular cross-section duct (6x896).

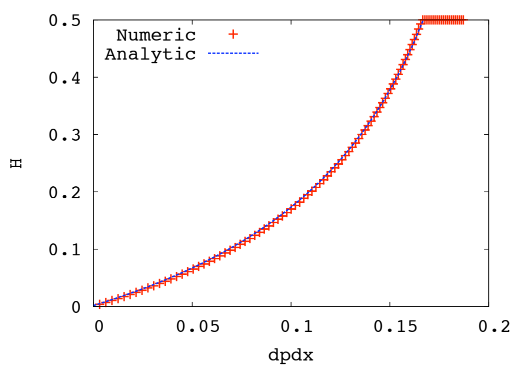


(a) Square duct

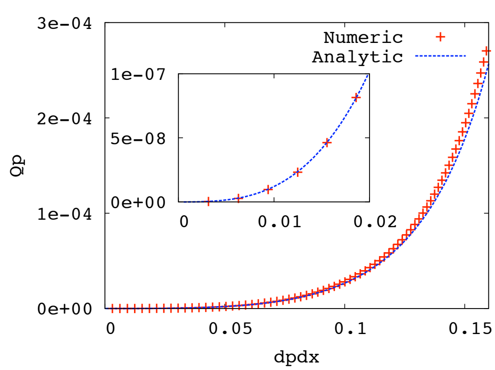
(b) Cylindrical duct

Figure 5: Velocity profile obtained by numerical simulations with the two-fluid model for a) the square cross-section duct (6x20x40) and b) the circular cross-section duct (6x896). The fluid phase velocity is in blue and the particulate phase velocity is in red. An offset of 10^{-3} has been added to the velocity of the particulate phase (up) to make it visible.

Figure 6 shows the application of the mixed-fluid model coupled with an Arc Length Continuation algorithm that allows us to explore the parameter space in terms of the longitudinal pressure gradient $\partial p/\partial x$. The agreement between the analytical solution of Ouriemi et al. (2009a) and the numerical model for the height of the flowing granular layer $H_p - H_c$ and the particle flux Q_p for a wide range of longitudinal pressure gradient gives another validation of our three dimensional numerical model. As for the velocity profiles we have explored the parameter space for three dimensional configurations that allows us to obtain the height evolution of the flowing granular layer $H_p - H_c$ and the particle flux Q_p in rectangular and circular cross section ducts. These results are presented in Figure 7 and illustrate the non-trivial effect of the geometry on the behaviour of the flowing granular layer. This method applied to complex geometry is a powerful tool to compare the prediction of the initial two-phase model with experiments in real geometry such as rectangular or cylindrical ducts. This is a strong argument for the development of a three dimensional numerical model.



(a) Height of the flowing layer



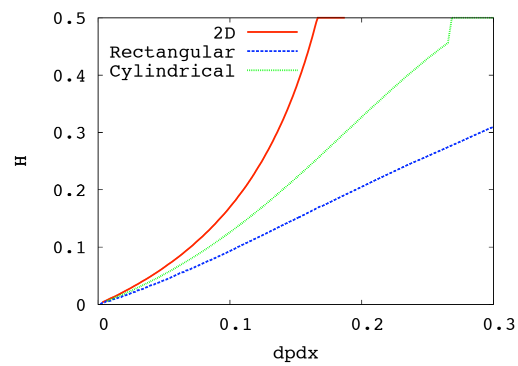
(b) Particle flux

Figure 6: Comparison of the numerical results for the 2D (mesh size 4x1x640) configurations with the analytical solution of Ouriemi et al. (2009a) in terms of a) the height of the flowing granular layer $H_p - H_c$ and b) the particle flux Q_p versus the longitudinal pressure gradient.

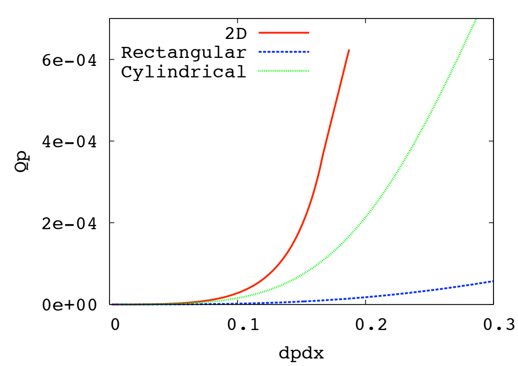
Conclusions

In summary, we have developped a three dimensional numerical model to simulate incompressible particulate two-phase flows for the flow of a Newtonian fluid over a granular bed. The incompressibility is imposed by a penalisation method and the viscoplastic characteristic of the granular rheology is dealt with a regularisation technique. We have studied the accuracy of our numerical solution by comparison with the analytical solution derived by Ouriemi et al. (2009a) in function of the spatial discretisation and the value of the regularisation parameter for the mixed-fluid and the two-fluid model. We have concluded that one must take a regularisation parameter one order of magnitude lower for the mixed-fluid model than for the two-fluid one to reach the same accuracy.

The Finite Element method allows us to deal with arbitrary geometries and we have shown numerical results



(a) Height of the flowing layer



(b) Particle flux

Figure 7: Comparison of the numerical results for the 2D and 3D configurations (rectangular and circular cross section ducts) in terms of a) the height of the flowing granular layer $H_p - H_c$ and b) the particle flux Q_p versus the longitudinal pressure gradient. The mesh sizes for the 2D case is 4x1x640, for the rectangular duct (aspect ratio of $W/H = 0.2692$) is 1x160x320 and for the cylindrical duct is 1x179998.

of three-dimensional simulation of the bed-load transport in a square and a circular cross-section ducts. This model had also been coupled with an Arc Length Continuation algorithm to describe the evolution of the height of the flowing granular layer and the particle flux with the longitudinal pressure gradient in both 2D and 3D configurations. This method is particularly interesting in the perspective of comparing the prediction of this model with experiments that are always carried out in 3D configurations.

Ouriemi et al. (2009b, 2010) have performed a simple linear stability analysis which provides realistic predictions for the formation of small dunes. The aim of the present numerical model is to perform a full three-dimensional stability analysis that accounts for the two-phase nature of the problem.

Acknowledgement

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